Soliton scattering from a finite cnoidal wave train in a fiber

H. J. Shin*
Department of Physics and Research Institute of Basic Sciences, Kyunghee University, Seoul 130-701, Korea
(Received 17 August 2000; published 22 January 2001)

We analyze the scattering of a soliton from a cnoidal wave train in a fiber theoretically as well as numerically. Solitons recover their original shapes and velocities after collisions, while shapes of cnoidal waves are nearly preserved during collisions. The effect of collisions is described by the change of velocities of solitons, and the theoretical predictions are in good agreement with numerical results.

DOI: 10.1103/PhysRevE.63.026606 PACS number(s): 42.65.Tg, 05.45.Yv, 05.60.Gg

There exist many interesting optical devices that incorporate soliton phenomena in optical materials. For instance, the time-domain chirp switch [1] as well as the soliton-dragging and soliton-trapping logic gates [2] are important examples of soliton applications. To support these technologies it is needed to understand theoretically collisions between solitons or between solitons and other wave packets. Up to now, main theoretical efforts were concentrated on scatterings between solitons, leading to the estimation of phase and time shift in elastic collisions. They are then applied to devise reversible optical logic gates, etc [3]. Contrary to elastic collisions between solitons, scatterings between solitons and other finite wave trains remain to be seriously investigated. It is partially due to the difficulty of theoretical analysis compared to the elastic scattering. But soliton collisions with finite wave trains seem to have important applications in various situations and have been on the forefront of active researches in some cases [4,5].

In this respect, recent theoretical analysis on soliton collisions with continuous wave (CW) trains of finite width [6] was a new attempt on this type of problems. This work, based on the existence of nonlinearly superposed solutions of the soliton and the infinite continuous wave in the nonlinear Schrödinger (NLS) equation, calculates the change of the velocity of a soliton when it encounters a finite CW train. The predictions were in good agreement with numerical experiments. It was also found that solitons restore their original shapes and velocities after the scatterings, even though they somewhat lose their identities during the scattering as they exchange energies periodically with CW light. Despite their intrinsic instabilities, finite CW lights were also found to maintain their identities more or less.

It is then of important necessity to find new collision processes between solitons and other finite wave trains that are manageable to theoretical and numerical analyses. In this paper we extend the previous results of [6] to those of collisions between the soliton and the cnoidal wave (CNW), another famous solution of the NLS equation [7]. For this, we construct the nonlinearly superposed solution of the soliton and the cnoidal wave (of infinite width), and perform numerical analyses on collisions between the soliton and the CNW train having finite width. The conclusion of our analysis can be summarized by the following three facts:

(1) A superposed solution of the soliton and the cnoidal wave is constructed using the Darboux-Bäcklund transformation (DBT). Solutions as well as important physical parameters like the DBT parameter are explicitly expressed using the Jacobi theta functions.

(2) Numerical analyses show that collision processes preserve the soliton characteristics when the amplitude of the finite CNW train is less than some proper value. The main effect of the collisions is then ascribed to the change of velocity of solitons during collisions.

(3) The DBT parameter of the nonlinearly superposed solution is found to remain unchanged during collision processes. This fact gives a relationship between the velocity of a soliton during a collision process with that of a finite CNW train.

All these results are coincident with those of Ref. [6], which can be thought of as a special case of this paper. We might expect that these aspects of collisions will be persistent for more general collisions. But a lack of theoretical tools to analyze these situations is the main obstacle in understanding these collisions.

The propagation of light waves in a fiber is described by the following NLS equation:

$$i\bar{\partial}\psi+\partial^2\psi+2|\psi|^2\psi=0, \tag{1}$$

where $\partial=\partial/\partial z$, $\bar{\partial}=\partial/\partial\bar{z}$, and $\bar{z}\equiv x$ and $z\equiv t-x/v_g$ represent the distance of propagation along the fiber and the (retarded) time. The NLS equation has the following cnoidal wave solution:

$$\psi_c(z,\bar{z})=p\,\mathrm{dn}(\chi,k)e^{i\zeta}, \tag{2}$$

where $\chi=p(z-v\bar{z})$, $\zeta=[vz/2+p^2(2-k^2)\bar{z}-v^2\bar{z}/4]$ and dn is the standard Jacobi elliptic function. Here v is the velocity of the cnoidal wave and $k\in(0,1)$ is the modulus of the Jacobi function. As far as elliptic functions are involved, we employ terminology and notation of Ref. [8] without further explanations. To obtain a superposed solution of a soliton and a cnoidal wave [9], we need to first find a solution of the following linear equations associated with the NLS equation,

*Email address; hjshin@khu.ac.kr

$$\partial\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} i\lambda & \psi_c(z,\bar{z}) \\ -\psi_c(z,\bar{z})^* & -i\lambda \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

$$\bar{\partial}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} -2i\lambda^2 + i|\psi_c(z,\bar{z})|^2 & -2\lambda\,\psi_c(z,\bar{z}) + i\partial\psi_c(z,\bar{z}) \\ 2\lambda\,\psi_c(z,\bar{z})^* + i\partial\psi_c(z,\bar{z})^* & 2i\lambda^2 + i|\psi_c(z,\bar{z})|^2 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \tag{3}$$

Then a solution that describes the superposition of a soliton and a cnoidal wave can be constructed using the Darboux-Bäcklund transformation [10]

$$\psi_{c-s}(z,\bar{z}) = \psi_c(z,\bar{z}) + 2i(\lambda - \lambda^*)\left(\frac{s_2}{s_1} + \frac{s_1^*}{s_2^*}\right)^{-1}. \tag{4}$$

Using the fact that s_i's satisfy the associated linear equations in Eq. (3), it can be explicitly checked that ψ_{c-s} in Eq. (4) is a solution of the NLS equation. Explicitly, the solution of the linear equations in Eq. (3) can be written down as [11]

$$\begin{aligned} s_1 &= e^{i\zeta/2}\left[e^{i\Delta}\theta_2\left(\frac{-iu}{2K}\right)\theta_0\left(\frac{\chi+iu}{2K}\right) \right. \\ &\quad \left. - Me^{-i\Delta}\theta_1\left(\frac{-iu}{2K}\right)\theta_3\left(\frac{\chi-iu}{2K}\right)\right] \Big/ \ \theta_0\left(\frac{\chi}{2K}\right), \\ s_2 &= e^{-i\zeta/2}\left[-e^{i\Delta}\theta_1\left(\frac{-iu}{2K}\right)\theta_3\left(\frac{\chi+iu}{2K}\right) \right. \\ &\quad \left. + Me^{-i\Delta}\theta_2\left(\frac{-iu}{2K}\right)\theta_0\left(\frac{\chi-iu}{2K}\right)\right] \Big/ \ \theta_0\left(\frac{\chi}{2K}\right), \end{aligned} \tag{5}$$

where M is an arbitrary complex number and $\Delta = \gamma\bar{z} + \beta\chi$ with

$$\gamma = -\frac{p^2}{2}\left[\mathrm{dn}^2(u,k') + \frac{\mathrm{cn}^2(u,k')}{\mathrm{sn}^2(u,k')}\right],$$

$$\begin{aligned} \beta &= iE\{\sin^{-1}[\mathrm{sn}(iu,k)]\} + \frac{E}{K}u + \frac{1}{2}\,\frac{\mathrm{dn}(u,k')\mathrm{cn}(u,k')}{\mathrm{sn}(u,k')} \\ &\quad + \frac{\mathrm{sn}(u,k')\mathrm{dn}(u,k')}{\mathrm{cn}(u,k')}, \end{aligned} \tag{6}$$

and K and E are complete elliptic integrals of the first and the second kinds, respectively. In the above formula, u is related to the DBT parameter λ as follows:

$$\lambda = \frac{v}{4} + \frac{p}{2}\,\frac{\mathrm{dn}(u,k')\mathrm{cn}(u,k')}{\mathrm{sn}(u,k')}. \tag{7}$$

When we explicitly substitute the result of Eq. (5) into Eq. (4), we can obtain a superposed solution of a soliton and a cnoidal wave. Figure 1 shows such an example, which we plot with typical parameters. The above result is reduced to the case of the collision of a soliton with a CW wave when we take $k \to 0$ (this is the case treated in [6]), and to that of scattering of two solitons for $k \to 1$.

Equations (4) and (5) show that the soliton in a cnoidal wave moves along the path $\mathrm{Im}\,D = \mathrm{Im}(\gamma\bar{z} + \beta\chi) = \mathrm{const}$, such that the velocity of the soliton becomes

$$v_{sol} = -\left(\frac{\mathrm{Im}\,\gamma}{\mathrm{Im}\,\beta p} - \frac{1}{v_g} - v\right)^{-1}. \tag{8}$$

To describe the collision of a soliton with a finite CNW train, we need to find the soliton characteristics when it is outside of the wave train. For this, we take the amplitude of the cnoidal wave in Eq. (4) to be zero, i.e. $p \to 0$. But ψ_{c-s} becomes singular unless we simultaneously take $u \to 0$. In fact, due to the following reason we take $p,u \to 0$ with their ratios fixed by the constraint $\lambda \equiv -(w_s + iA_s)/2 = \mathrm{const.}$ As we can see in Fig. 3, numerical experiments show the amplitude of the soliton, i.e. $\lambda - \lambda^*$ in Eq. (4) is almost preserved during the collision. Theoretically, it is also reasonable to take the DBT parameter λ conserved as it is a physical parameter describing the amplitude and the velocity of a soliton. Actually this fact is already used in [6] to successfully describe the soliton and CW wave scattering.

For the simplicity of discussions, we take $v = 0, v_g = \infty$ hereafter. Then in the limit of $(p,u \to 0)$, Eq. (7) becomes $u = -p/(w_s + iA_s)$, and Eq. (4) becomes

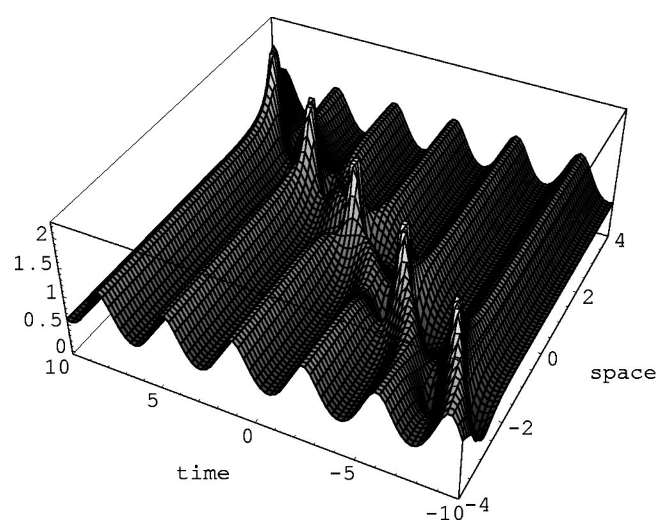

FIG. 1. Intensity profile (theoretical) of soliton + infinite CNW train with parameters $A_s = 1.84$, $w_s = -0.7$, $p = 1.3$, $k = 0.9$, $v = 0$, and $u = -0.38 + 0.63i$.

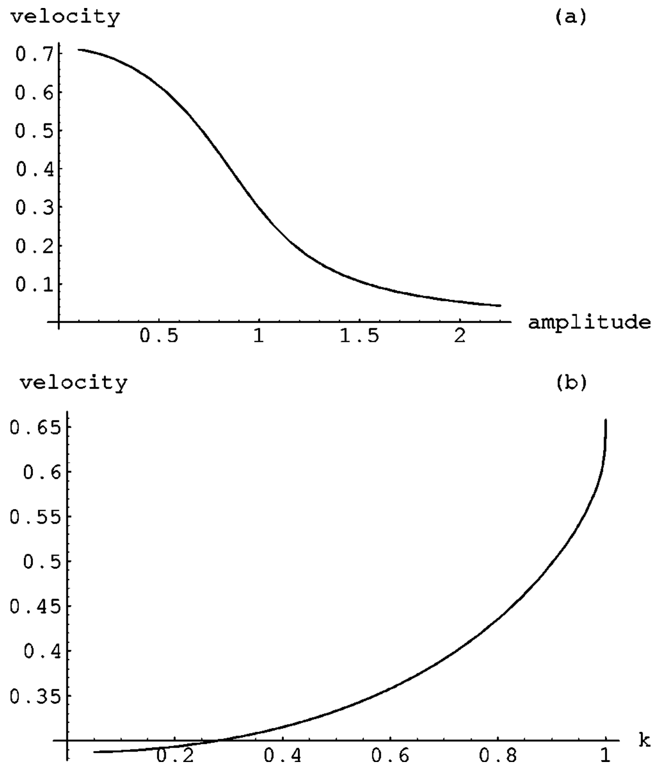

FIG. 2. (a) Soliton velocity vs CNW amplitude p for $k=0.24$, (b) soliton velocity vs CNW modulus k for $p=1$, with $A_s=1.84$, $w_s=-0.7$.

$$\psi_{c-s}=A_s\ \mathrm{sech}[A_s z+2w_sA_s\overline{z}]\exp[i\{-w_s z-(w_s^2-A_s^2)\overline{z}\}], \tag{9}$$

which describes a soliton with amplitude A_s and frequency w_s. On the other hand, when we take $\lambda-\lambda^*=-iA_s\rightarrow 0$ in Eq. (4), it becomes the cnoidal wave ψ_c.

Now, the velocity of a soliton inside the CNW train can be expressed as a function of A_s, w_s, p, and k using Eq. (8) where we take $u=u(A_s,w_s,p,k)$ as a solution of Eq. (7) for a given soliton characteristic $\lambda=-(w_s+iA_s)/2$. When the soliton resides outside the CNW train, the velocity becomes $-(2w_s)^{-1}$, which is obtained using Eq. (9) or by taking $p,u\rightarrow 0$ on the soliton velocity inside the CNW train. Figure 2 shows the velocity of a soliton inside a CNW train for typical values of A_S, w_s, p, and k [12]. It shows that the impedance effect of a CNW train on a soliton is smaller than that of a CW train.

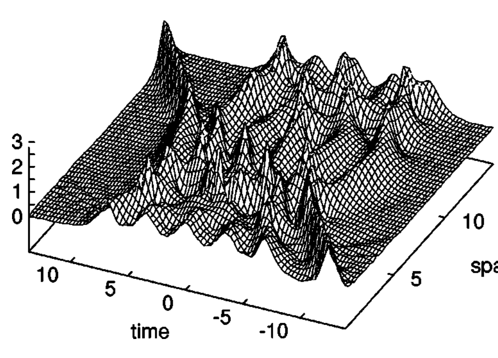

FIG. 3. Intensity profile (numerical) of soliton + finite CNW train with the same parameters as in Fig. 1.

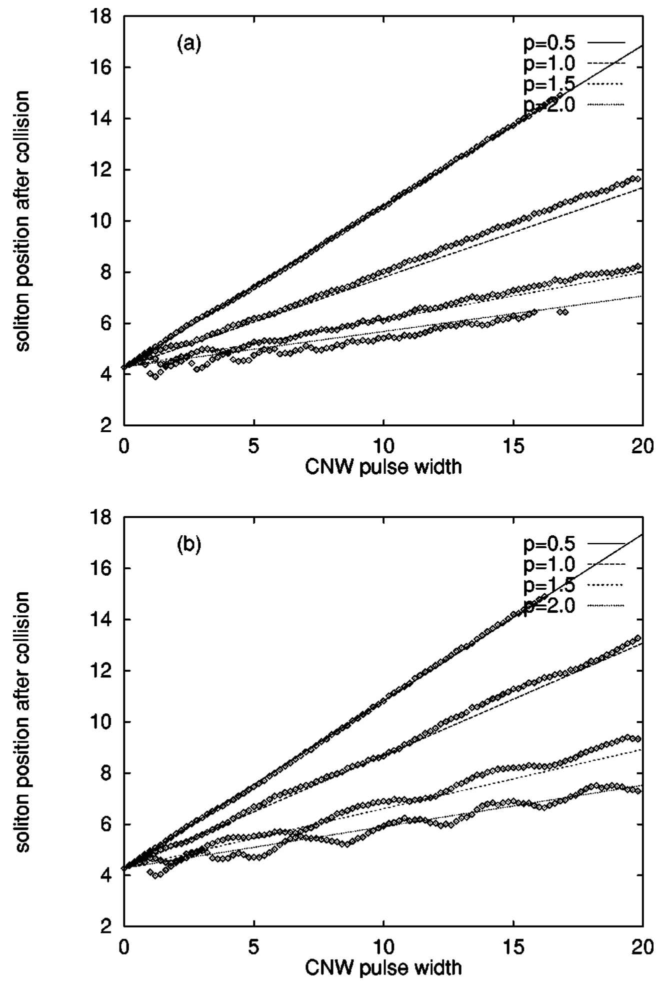

FIG. 4. Position shift vs width of CNW train for amplitudes $p=0.5,1,1.5,2$. (a) $k=0.24$, (b) $k=0.72$ with $A_s=1.84$, $w_s=-0.7$, and $v=0$.

Figure 3 describes the collision of a soliton with a finite CNW train (with the same parameters as in Fig. 1) that was obtained using the split-step fast Fourier transform algorithm. The soliton injected into the right side of the CNW train reappears on the left side of the train after the collision. It can be clearly seen that the velocity of the soliton changes during collision. Numerical analyses show that this type of collision is maintained until the amplitude of the CNW train almost reaches that of the soliton, where the instability of the CNW train does not allow the identification of the soliton anymore.

The following numerical plots also confirm the above picture of a soliton collision with a CNW train. Figure 4 plots the moving distance $\Delta\overline{z}=\Delta x$ of a soliton along a time width of a CNW train, Δz. In numerical simulations, we took $2D=6$ as the time interval the soliton moves without encountering the CNW train and Δz (x axis in Fig. 4) as the interval for the soliton lying inside the CNW train. Thus, the total moving distance ($\Delta\overline{z}$, y axis in Fig. 4) is theoretically $\Delta\overline{z}$

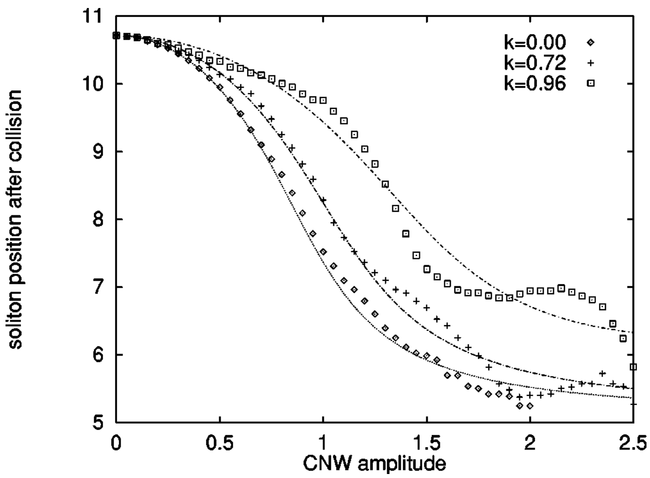

FIG. 5. Position vs CNW amplitude for $k=0,0.72,0.96$ with $A_s=1.84$, $w_s=-0.7$, and $v=0$.

$=-(2w_s)^{-1}2D+v_{sol}\Delta z$, which is also plotted in Fig. 4 [13]. Numerical results are well in accordance with theoretical plots for wide ranges of Δz and p, k. Especially, the linear dependence of the moving distance on $\Delta\overline{z}$ convinces us about the concept of the change of velocities of solitons inside CNW trains. Note that the effect of the collision of two solitons was described by the phase and time shift conventionally. The terms $\theta_i(\chi/2K)$ in Eq. (5) bend the curves in Fig. 4, especially for large values of k. Figure 5 plots the moving distance of a soliton along the amplitude p of CNW waves, where we take the time width as $\Delta z=9$. This figure also shows that the theoretical curves fit well with the numerical results without any fitting parameters.

All these figures show the validity of our picture for the soliton-CNW train collision. This description can be applied to the scattering of multisolitons with a CNW train too. In this case, each soliton experiences a change of velocity during the collision with a finite CNW train, while the CNW train itself suffers no essential change. Our description of the collision in terms of velocity changes could be a good starting point for an approximate description of collisions between various light-wave packets. Especially solutions expressed in terms of the DBT parameter will be very useful for this type of a description as it remains unchanged during collision processes. Possible extension of our method to more general collision phenomena might be achieved using a WKB-type approximation and is reserved for future study. The instability-dominating collision processes where solitons loose their identities would be another important and interesting research area to be investigated.

This work was supported by the Brain Korea 21 Project.

[1] See, for example, M. N. Islam, *Ultrafast Fiber Switching Devices and Systems* (Cambridge University Press, New York, 1992), and references therein.
[2] C. R. Menyuk, Opt. Lett. **12**, 614 (1987); J. Opt. Soc. Am. B **5**, 392 (1988).
[3] E. Fredkin and T. Toffoli, Int. J. Theor. Phys. **21**, 219 (1982).
[4] A. Hasegawa and Y. Kodama, Opt. Lett. **7**, 285 (1982).
[5] B. Jaskorzynska and D. Schadt, IEEE J. Quantum Electron. **24**, 2117 (1988); E. J. Greer, D. M. Patrick, P. G. J. Wigley, and J. R. Taylor, Opt. Lett. **15**, 851 (1990).
[6] Q-H. Park and H. J. Shin, Phys. Rev. Lett. **82**, 4432 (1999).
[7] J. Mertsching, Fortschr. Phys. **35**, 519 (1987).
[8] *Encyclopedic Dictionary of Mathematics*, edited by K. Itô (MIT Press, Cambridge, MA, 1993).
[9] F. Abdullaev, S. Darmanyan and P. Khabibullaev, *Optical Solitons*, Springer Series in Nonlinear Dynamics (Springer-Verlag, Heidelberg, 1993); their solutions through the Riemann surface have a rather complicated form that prevents their applications to real physical situations; see also, A. Kamchatnov, Phys. Rep. **286**, 199 (1997).
[10] Q-H. Park and H. J. Shin, Phys. Rev. A **57**, 4621 (1998); **57**, 4643 (1998); Phys. Rev. E **61**, 3093 (2000); Opt. Commun. **178**, 233 (2000).
[11] A. Sym, Fluid Dyn. Res. **3**, 151 (1988).
[12] A simple MATHEMATICA program for the velocity is available from the author.
[13] Like in Ref. [6], we use the effective time interval $D_{eff}=0.87\Delta\overline{z}$, $2D\rightarrow 2D+0.13\Delta z$ due to the rapid round off of square pulse edges of the CNW train.